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Absolute Stability of Noncompact Sets*

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1. INTRODUCTION

No doubt, most workers in dynamical system theory are aware that the customary concepts of stability, perfectly adequate for compact sets, are somehow badly wrong for sets which are closed but not compact. There is the trivial example of two asymptotically stable sets whose product is neither stable nor weakly attracting (e.g., the x axis for $\dot{x} = x$, $\dot{y} = -y$; this is not even semistable in the sense of [1]).

The stability concepts we have in mind are, e.g., ordinary, absolute and asymptotic stability. (For most of the definitions see [2, Chap. 2].) In the present paper we show that, somewhat surprisingly, the concept of absolute stability behaves well even for noncompact sets. By unwritten law, appropriateness of a stability concept is also judged by its amenability to a characterization in terms of Liapunov functions. For closed absolutely stable sets, this is our basic result, Theorem 15.

The tools needed are collected in Section 3. These consist, first, of an extension theorem for functions which are monotone in a suitable sense, yielding Liapunov functions in our applications. It is a consequence of a theorem due to Nachbin, applying to compact spaces; a not too special case was proved by Auslander [3, Theorem 4]. The second (Lemma 3) is, essentially, the decomposition theorem for paracompact locally compact spaces.

In Section 4 we characterize the sets $D_\infty^+(x)$ in terms of Liapunov functions. It will be seen that this is a logical extension of Auslander's results on the Liapunov-type description of points of generalized recurrence.

In Section 5 these results are used to obtain a Liapunov characterization of absolutely stable sets. These are rather strong extensions of known theorems applying to compact absolutely stable sets, e.g., [5, 4].

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Section 6 describes the absolutely stable sets in terms of properties of the class of all such subsets (of a given phase space).

Our results (principally, Theorem 15 and Corollary 16) appear to suggest that, to obtain definitions of various stability concepts which are to be useful even for noncompact sets M, formulations such as "for every neighborhood of M..." should be replaced by "M is the intersection of neighborhoods such that ...".

2. Terminology and Notation

For terminology and notation, see [2, Chap. 2], with modifications listed below. In particular, we will use the concepts of trajectories C_x, (first positive) prolongations $D_x^+ = D_1^+(x)$, (positive) invariance and stability, (positive) absolute stability, etc. If $\pi : X \times R^+ \to X$ is a dynamical system on a topological space X, then the value of π at a point $(x, t) \in X \times R^1$ will be denoted by $x\pi t$; and similarly for $A\pi B$, where $A \subset X$, $B \subset R^1$ (e.g., $C_x^+ = x\pi R^+$). In contrast to [2], our phase spaces X are rather general topological spaces. This entails some changes; e.g., the prolongational set D_x^+ or $D_1^+(x)$ is the set of all limits of nets (rather than sequences) $x_i \pi t_i$, where $x_i \to x$, $t_i \in R^+$. Nevertheless, the formula

$$D_x^+ = \bigcap \overline{U\pi R^+}$$

(intersection over all neighborhoods U of x) is preserved. Similarly, $D_x^+ = C_x^+ \cup J_x^+$ holds if X is a Hausdorff space.

The higher prolongations (see [4, 2, 5]) are best introduced in another manner. Given a dynamical system on X, consider the relation C^+, of "being on the positive trajectory of": xC^+y iff $x \in C_y^+$. Similarly, one defines the relations D^+, J^+ (and also, e.g., D^-, K^+, etc.). It is then easily checked that D^+ is precisely the closure of C^+ (as a subset of $X \times X$). Then one defines the second prolongation as the closure of the transitivization of D^+:

$$D_2^+ = \overline{D^{+T}}, \qquad D^{+T} = D^+ \cup D^+ \circ D^+ \cup D^+ \circ D^+ \circ D^+ \cup \cdots = \bigcup_{n=1}^{\infty} D^{+n}.$$

The higher prolongational relations are then obtained by an obvious induction process: if D_α^+ has been defined for all ordinals α, $1 \leqslant \alpha < \beta$, then let

$$D_\beta^+ = \text{closure}\left(\bigcup \{D_\alpha^{+T} : \alpha < \beta\}\right).$$

It is readily verified that $D^+ \subset D_2^+ \subset \ldots$, and that if $D_\alpha^+ = D_{\alpha+1}^+$, then actually $D_\alpha^+ = D_\beta^+$ for all $\beta > \alpha$. Thus there exists a least such ordinal α (its cardinality is at most that of the family of all subsets of $X \times X$); the corresponding prolongation D_α^+ is called the absolute prolongation, and denoted by D_∞^+ (in [4] it is shown that $\alpha \leqslant \omega_1$ in the case of metric spaces). From this construction it is obvious that D_∞^+ can alternately be characterized as the least closed transitive relation containing C^+. An analogous construction (beginning with J^+, not L^+) introduces J_α^+ and J_∞^+.

Now we may return, from the "prolongational relations", to the corresponding sets. For $x \in X$ (or $M \subset X$) and any ordinal $\alpha \geqslant 1$ (or $\alpha = \infty$) let

$$D_\alpha^+(x) = \{y : yD_\alpha^+x\}, \qquad D_\alpha^+(M) = \{y : yD_\alpha^+x \text{ for some } x \in M\}.$$

A mapping $v : X \to R^1$ on the phase space X of a dynamical system π is called a Liapunov function (for, or relative to, π) iff v is continuous and

$$v(x\pi t) \leqslant v(x) \qquad \text{for all} \quad x \in X, \quad t \geqslant 0.$$

Informally, the last condition may be formulated as "v decreases along trajectories". The meaning of phrases such as v strictly decreases along C_x, or v is constant along C_x, is perhaps obvious.

3. Topological Preliminaries

Let ρ be a relation on a set X (thus $\rho \subset X \times X$). A real-valued function $f : X \to R^1$ may be termed ρ-decreasing iff

$$f(x) \leqslant f(y) \qquad \text{whenever} \quad x\rho y. \tag{1}$$

It will be useful to apply the same term to partial functions $f : Y \to R^1$, $Y \subset X$, iff they satisfy (1) with both x, y in Y; actually, this is $\rho \cap (Y \times Y)$-decreasingness in the previous sense.

In the same situation, a subset $M \subset X$ may be termed positively ρ-invariant iff $x\rho y \in M$ always implies $x \in M$ (negatively ρ-invariant iff $M \ni x\rho y$ implies $y \in M$; this is positive ρ^{-1}-invariance).

As examples particularly significant for our purposes, a continuous function $f : X \to R^1$ on a phase space X is a Liapunov function iff it is C^+-decreasing (equivalently, D_∞^+-decreasing, etc.); a subset $M \subset X$ is positively invariant iff it is positively C^+-invariant (positively absolutely stable iff it is positively D_∞^+-invariant, etc.).

Theorem 1. *Let ρ be a closed, reflexive and transitive relation on a compact Hausdorff space X. Then every continuous ρ-decreasing function $f : M \to R^1$*

on a closed subset $M \subset X$ can be extended to a continuous ρ-decreasing $g : X \to R^1$ defined on the entire space X. (If, furthermore, $0 \leqslant f \leqslant 1$, then we may also require $0 \leqslant g \leqslant 1$.)

Proof. The assertion is a direct consequence of Theorem 3 in [6, p. 43]. We need to verify two assumptions there. First, that X with ρ is "normally preordered"; this follows from the definition (6, p. 28), using the fact that ρ is closed transitive and X compact Hausdorff; second, the condition appearing in the statement of the mentioned theorem. This is verified as in Theorem 6 of [6]; that here we are dealing with a "preorder" rather than an "order" has no effect.

Remark. Without compactness the assertion above is false, even for $X = R^2$. As an example, define ρ by letting $(x, y)\rho(x', y')$ iff $x \leqslant x'$, $y = y'$ (this is the relation C^+ for the obvious parallel system on R^2). Let $M = M_1 \cup M_2$ with M_1 the x-axis and $M_2 = \{(x, y) : |y| \geqslant e^{-x}\}$. Let $f : M \to R^1$ be 0 on M_1 and 1 on M_2. Evidently, there is no ρ-decreasing extension of f which is continuous on M_1.

However, under further assumptions, compactness of X may be relaxed considerably.

THEOREM 2. *Let ρ be a closed, reflexive and transitive relation on a space X which is Hausdorff, σ-compact and locally compact. Let $f : M \to [0, 1]$ be a continuous ρ-decreasing function on a compact subset $M \subset X$. Finally, assume given closed sets P, Q, such that P, M, Q are disjoint, P is positively ρ-invariant, Q is negatively ρ-invariant.*

Then there exists a continuous ρ-decreasing function $g : X \to [0, 1]$ such that

$$g(x) = \begin{cases} 0 & \text{for } x \in P \\ f(x) & \text{for } x \in M \\ 1 & \text{for } x \in Q. \end{cases}$$

Proof. By assumption on X, there is a sequence of sets X_n such that

$$X = \bigcup_{n=1}^{\infty} X_n, \qquad X_n \text{ compact}, \qquad X_n \subset \operatorname{int} X_{n+1}.$$

Since M is compact (hence covered by finitely many X_n's), we may even assume that $M = X_1$.

Set $f_1 = g_1 = f$. Define f_2 by letting

$$f_2(x) = \begin{cases} 0 & \text{for } x \in P \cap X_2 \\ g_1(x) & \text{for } x \in X_1 \\ 1 & \text{for } x \in Q \cap X_2. \end{cases}$$

Evidently, f_2 is continuous, and maps a closed subset of X_2 into [0, 1]. In order to apply Theorem 1 (to f_2 and X_2), we need to check that f_2 is ρ-decreasing. This follows from ρ-decreasingness of $g_1 = f$, from the ρ-invariance assumptions on P, Q, and from $0 \leqslant g_1 \leqslant 1$.

Thus there is a continuous ρ-decreasing extension of g_2 of f_2, $g_2 : X_2 \to [0, 1]$, and g_2 satisfies (2). Apparently the construction may be continued inductively, yielding a mapping g from $\bigcup_{n=1}^{\infty} X_n = X$ into [0, 1], and satisfying (2). Continuity of g follows from $X_n \subset \operatorname{int} X_{n+1}$, that it is ρ-decreasing from transitivity of ρ.

Remarks. A similar proof yields the following assertion. With the same assumptions on X, let $f : M \cup P \to R^1$ be continuous and ρ-decreasing, M compact, P closed and both positively and negatively ρ-invariant. Then there exists a continuous ρ-decreasing extension g of f, $g : X \to R^1$.

The next result will enable us to avoid the assumption of σ-compactness on X. However, we restrict the relations ρ involved to our dynamical relations C^+, $D_\infty{}^+$, etc. (Actually, the only property required is that, if $x\rho y$, then both x, y are in the same quasicomponent of X.)

REDUCTION LEMMA 3. *Let π be a dynamical system on a space X which is Hausdorff, paracompact and locally compact. Then X is the direct sum of spaces X_i which are σ-compact (and Hausdorff locally compact). Furthermore,*

3.1. *Each X_i is invariant and bilaterally absolutely stable; $D_\infty(x) \subset X_i$, for each $x \in X_i$; in particular, there is a relativitation π_i of π over X_i.*

3.2. *A set $M \subset X$ is closed, or absolutely stable, or asymptotically stable, iff each $X_i \cap M$ is such, relative to π_i.*

3.3. *A function $v : X \to R^1$ is a Liapunov function iff each partialization $v_i = v \mid X_i$ is such, relative to π_i.*

Proof. The direct sum decomposition is well known: [7, XI, Theorem 7.3]. Invariance of the X_i follows from the fact that, in each decomposition $X = Y_1 \cup Y_2$ into disjoint open sets, both Y_k are invarient. This also yields $D^+(Y_i) = Y_i$, whereupon $D_\alpha{}^+(Y_i) = Y_i$ for all ordinals α is obtained by a simple induction.

Invariance and openness of the X_i then yields assertions 3.2 and 3.3 directly.

4. $D_\infty{}^+$ AND LIAPUNOV FUNCTIONS

LEMMA 4. *If $x \in D_\infty{}^+(y)$, then $v(x) \leqslant v(y)$ for every Liapunov function v.*

Proof. This can be carried out by the obvious induction; however, there is a more elegant proof. Introduce a relation V on X by letting xVy iff

$v(x) \leqslant v(y)$ for every Liapunov function v. Evidently, V is closed transitive, and contains the relation C^+ (since Liapunov functions decrease along trajectories). But D_∞^+ is the smallest closed transitive relation containing C^+; thus $D_\infty^+ \subset V$, which is our assertion.

THEOREM 5. *Let π be a dynamical system on a space X which is Hausdorff, paracompact and locally compact. Then $x \in D_\infty^+(y)$ if and only if $v(x) \leqslant v(y)$ for every Liapunov function v on X.*

Proof. Having Lemma 4, we only need to show that, if $x \notin D_\infty^+(y)$, then $v(x) = 1$, $v(y) = 0$ for some Liapunov function v. Invoke the Reduction Lemma 3. If x, y are in distinct direct summands X_i, e.g., $x \notin X_i \ni y$, then define $v : X \to [0, 1]$ as 0 on X_i and 1 elsewhere; according to 3.3, v is a Liapunov function. If both x, y are in the same direct summand X_i, first define v as 0 outside X_i; to define it on the σ-compact space X_i, proceed thus.

First, let $w(x) = 1$, $w(y) = 0$. Evidently, w is then a continuous D_∞^+-decreasing function on the compact set $\{x\} \cup \{y\}$. According to Theorem 2 (with $P = Q = \varnothing$) there is a continuous D_∞^+-decreasing extension v of w to the entire space X_i. In particular, v is C^+-monotone, i.e., it decreases along trajectories, as required.

We invert Auslander's definition of generalized recurrence [3, p. 66, and Theorem 3]:

DEFINITION 6. A point x is termed a point of generalized recurrence iff $x \in J_\infty(x)$.

Evidently critical points, periodic points, nonwandering points are all points of generalized recurrence (use $J^+ \subset J_\infty^+$). In particular, each point in some limit set is such (since $x \in L_y^+$ implies $x \in J_x^+$). Thus Poisson stable points, and also points in the center of the dynamical system are points of generalized recurrence.

LEMMA 7. 7.1. *The following are mutually equivalent: $x \in J_\infty(x)$, $x \in J_\infty^+(x)$, $x\pi t \in D_\infty^+(x\pi s)$ for all t, s, $x \in D_\infty^+(x\pi s)$ for some $s > 0$.*

7.2. *If x is a point of generalized recurrence, and v is any Liapunov function, then v is constant along C_x.*

Proof. 7.1 is essentially contained in [3, Lemma 5]; 7.2 follows hence and from Lemma 4.

THEOREM 8. *Let π be a dynamical system on a space X which is Hausdorff, paracompact and locally compact. Then $x \in X$ is a point of generalized recurrence iff every Liapunov function for π is constant along C_x.*

Proof. One part follows from 7.2, the other from Theorem 5 and 7.1.

THEOREM 9. *Let A be the (closed) set of all points of generalized recurrence in a phase space X which is Hausdorff, paracompact and locally compact. Then A is a G_δ set iff there exists a Liapunov function v on X with the following properties: for any $x \in X$,*

9.1. *v is constant on C_x if $x \in A$.*

9.2. *v strictly decreases along C_x if $x \notin A$.*

Proof. This will be split up into several steps.

9.3. If such a test-function v exists, then A coincides with

$$\bigcap_{m=-\infty}^{\infty} \{x : v(x) = v(x\pi m)\} = \bigcap_{m=-\infty}^{\infty} \bigcap_{n=1}^{\infty} \left\{x : v(x) - \frac{1}{n} < v(x\pi m)\right\},$$

which is obviously a G_δ set.

9.4. Conversely, let $A = \cap G_m$, G_m open; according to Lemma 3 we may assume that $X = \cup X_n$, X_n compact. From our assumptions on X, the space $\mathbf{C}(X)$ is complete metric (this is the set of all continuous functions $X \to R^1$, endowed with the compact-open topology). Let L consist of all Liapunov functions $X \to R^+$; evidently L is closed in $\mathbf{C}(X)$, so that it is also complete and hence a Baire space.

9.5. For any compact set $W \subset X$, define

$$L_W = L(W) = \{v \in L : v(x) > v(x\pi 1) \text{ for all } x \in W\}.$$

We assert that L_W is open in L. Indeed, if not, there exist $v_i \in L$ with

$$L_W \not\ni v_i \to v \in L_W ,$$

so that there are $x_i \in W$ with $v_i(x_i) \leqslant v_i(x_i\pi 1)$. Since W is compact, we may take a convergent subset $x_j \to x \in W$; and then, taking limits, we obtain $v(x) \leqslant v(x\pi)1$, contradicting $v \in L_W$ and $x \in W$.

9.6. Every point $x \notin A$ has a compact neighborhood $W = W(x)$, such that L_W is dense in L. To construct W, first conclude $x \notin D_\infty^+(x\pi 1)$ from 7.1 and apply Theorem 5 to obtain a Liapunov function w_0 with $w_0(x) > w_0(x\pi 1)$; apparently, we may as well assume that $w_0(x) > 1 > 0 > w_0(x\pi 1)$. From continuity, this inequality also holds for points in a (compact) neighborhood W of x. Set $w = \min(1, \max(w_0, 0))$; thus w is a Liapunov function $X \to [0, 1]$ with values 1 in W, 0 in $W\pi 1$. Finally show that L_W is dense in L. Take any $v \in L$; then, for each $\epsilon > 0$, v is ϵ-close to $v + \epsilon w$, and, obviously, $v + \epsilon w \in L_W$.

9.7. Consider any combination of indices n, $m \geqslant 1$. The set $X_n - G_m$

is compact and disjoint with A. From 9.5 and 9.6, there is a finite cover of $X_n - G_m$ by sets W such that L_W is open and dense in L. Thus there is a countable cover of

$$\bigcup_{nm} (X_n - G_m) = \bigcup X_n - \bigcap G_m = X - A$$

by such sets W_k. Since L is a Baire space, $\cap L(W_k)$ is nonvoid, and hence contains some Liapunov functions. We observe that every $x \notin A$ is in some W_k, and so $w(x) > w(x\pi 1)$ from $w \in L(W_k)$. In particular, w has property 1; to obtain the second property define a new Liapunov function v by

$$v(x) = \int_0^1 w(x\pi t)\, dt.$$

It is easily verified that v strictly decreases along C_x for $x \notin A$. This completes the proof.

COROLLARY 10. *If X is metrizable locally compact, then there exists a Liapunov function with properties* 9.1 *and* 9.2.

Remarks 11. 11.1. The strength of Theorem 5 may be suggested by the following. In terms of the relation V from the proof of Lemma 4, the assertion is that $D_\infty^+ = V$. Now, it can be shown that, whenever $x \in D_\infty^+(y)$ and M is a set, with compact boundary, containing y but not x, there exists an "intermediate" point u with

$$x \in D_\infty^+(u), \qquad u \in D_\infty^+(y), \qquad u \in \partial M.$$

(This property is a substitute for connectedness of the sets $D_\infty^+(y)$.) However, a similar result does not seem to be directly provable for the relation V, practically equivalent with D_∞^+.

11.2. It should be emphasized that Auslander actually proved Theorem 5, but stated only Theorem 8 (with added assumptions on the phase space X). The references needed for Theorem 1 were also supplied to the author by Prof. Auslander.

11.3. Theorem 9 for separable locally compact metric spaces is the content of Theorems 2 and 3 in [3]. Here separability may be omitted via the reduction lemma. However, Auslander's proof depends strongly on separability of $C(X)$; and thus, for spaces X as in our theorem, it is actually equivalent to metrizability (the last assertion is a consequence of [8, 7S(d)], using the fact that a paracompact locally metrizable space is metrizable). Our proof thus had to proceed in another manner, i.e., via the Baire theorem.

11.4. In Theorem 9 we may also assume that $0 \leqslant v \leqslant 1$. However, we cannot assert that on each trajectory C_x individually the values of v cover the interval (0, 1) (see 11.5). Actually, the latter would imply parallelizability: indeed, for any $a < b$, on the open invariant subset

$$X_1 = \{x \in X : (a, b) \subset v(C_x)\},$$

the dynamical system is parallelizable, since it has $\{x \in X_1 : v(x) = \frac{1}{2}(a + b)\}$ as a global section.

11.5. For an example, begin with the parallel system on R^2 defined by $\dot{x} = 1, \dot{y} = 0$, and introduce a single critical point at the origin. Then, for the system restricted to $X = R^2 - \{0\}$, the function v defined by $v(x, y) = x$ is a test function; there are no points of generalized recurrence, but the system is not dispersive.

5. Absolute Stability

The following definition is due to Taro Ura [5] (see also [4, 2]).

Definition 12. A subset M of a phase space is called (positively) absolutely stable iff $D_\infty{}^+(M) = M$; similarly for negative and bilateral absolute stability.

It is known that asymptotic stability implies absolute stability, under very weak conditions; of course, absolute stability implies α-stability for $\alpha = 1$ (i.e., $D^+(M) = M$), and thus implies ordinary stability for compact sets in locally compact spaces. Evidently $D_\infty{}^+(x)$ is absolutely stable for every $x \in X$.

The first result of the following lemma follows directly from the definition; the second is a consequence of Lemma 4.

Lemma 13. 13.1. *The intersection of absolutely stable sets is absolutely stable.*

13.2. *If $v : X \to R^1$ is a Liapunov function, then all sets of the form $v^{-1}(-\infty, a]$, $a \in R^1$, are closed and absolutely stable.*

The following result will not be needed subsequently; it pertains directly to a remark in our introduction.

Corollary 14. *Arbitrary products of absolutely stable sets are absolutely stable.*

Proof. For $i = 1, 2$, let M_i be an absolutely stable subset of a phase space X_i. It is easily seen that $M_1 \times X_2$ is absolutely stable; from 13.1, so is

$$M_1 \times M_2 = (M_1 \times X_2) \cap (X_1 \times M_2).$$

Similarly, for any number of factor spaces.

THEOREM 15. *Let M be a subset of a space X which is Hausdorff, paracompact and locally compact. Then M is closed and absolutely stable iff $M = \cap v_i^{-1}(0)$ for suitable Liapunov functions $v_i : X \to [0, 1]$.*

Proof. That such an intersection is absolutely stable follows from Lemma 13. Conversely, let M be closed absolutely stable, and X σ-compact (see the Reduction Lemma 3). Consider any point $x \notin M$. According to Theorem 2, there exists a Liapunov function $v_x : X \to [0, 1]$ with $v_x(x) = 1$, $v_x = 0$ on M (as the three sets P, M, Q choose M, $\{x\}$, $\varnothing$, respectively). Evidently,

$$M = \bigcap \{v_x^{-1}(0) : x \notin M\}.$$

COROLLARY 16. *With the same assumptions on X, every closed absolutely stable set is the intersection of closed absolutely stable and G_δ neighborhoods.*

Proof. The sets $v_i^{-1}[0, a]$ are closed absolutely stable neighborhoods of M for $a > 0$; evidently each is a G_δ-set.

THEOREM 17. *Let M be a subset of a phase space X which is Hausdorff, paracompact and locally compact. Then the following properties of M are mutually equivalent:*

17.1. *$M = v^{-1}(0)$ for some Liapunov function $v : X \to [0, 1]$.*

17.2. *M is the intersection of a sequence of closed absolutely stable neighborhoods of M.*

17.3. *M is closed, absolutely stable and a G_δ-set.*

Proof. Referring to Lemma 13, we see that $1 \Rightarrow 2 \Rightarrow 3$. To prove $3 \Rightarrow 1$ let $M = \bigcap_{n=1}^{\infty} G_n$ be closed absolutely stable, G_n open; according to Lemma 3, we may assume that $X = \bigcup_{m=1}^{\infty} X_m$ with X_m compact. From Theorem 8, for any $x \notin M$, there is a Liapunov function $v : X \to [0, 1]$ with $M \subset v^{-1}(0)$, $v(x) > 0$. Here we may even assume that $v = 1$ in a neighborhood of x (replace V by $\epsilon^{-1} \min(v, \epsilon)$ with $\epsilon = \frac{1}{2}v(x)$). Now consider the compact set $X_m - G_n$ for any combination of indices m, n; it is covered by a finite system of such neighborhoods; taking maxima we obtain a Liapunov function $v_{nm} : X \to [0, 1]$ such that

$$M \subset v_{nm}^{-1}(0), \qquad X_m - G_n \subset v_{nm}^{-1}(1).$$

Set $v = \sum_{n,m=1}^{\infty} 2^{-(n+m)} v_{nm}$; this is a Liapunov function $X \to [0, 1]$ with $M \subset v^{-1}(0)$. To show that the inclusion is an equality, note that every point $x \notin M$ is in some $X_m - G_n$, whereupon $v_{nm}(x) > 0$.

Corollary 18. *In a locally compact metric phase space X, the closed absolutely stable sets are precisely the zero-sets of Liapunov functions $X \to [0, 1]$.*

Corollary 19. *In a locally compact metric phase space X, the following conditions on a closed subset M are equivalent:*

19.1. *M is absolutely stable and isolated from points of generalized recurrence (i.e., there are no such points in $U - M$ for some neighborhood U of M).*

19.2. *There exists a Liapunov function $v : X \to R^+$ with $M = v^{-1}(0)$ which strictly decreases along trajectories near M; more precisely, there is a positively invariant neighborhood V of M such that*

$$v(x) > v(x\pi t) \qquad \text{if} \quad t > 0, \quad x \in V, \quad x\pi t \notin M.$$

Proof. $2 \Rightarrow 1$ from Theorems 17 and 8. For the converse, set $v = v_1 \cdot v_2$, where $v_1 : X \to [0, 1]$ is a Liapunov function with zero-set M (Theorem 17) and v_2 is a test-function for generalized recurrence (Corollary 10); here, evidently, we may assume that $0 < v_2 < 1$.

Remark. In 19.2 we cannot assert that $v(x\pi t) \to 0$ as $t \to \infty$ for $x \in U$. Example: For the parallel system on R^2 ($\dot{x} = 1$, $\dot{y} = 0$), introduce critical points at $z_n = (x_n, y_n)$, $x_n \to +\infty$, $0 < y_n \to 0$, and then omit the open half rays $(x_n, +\infty) \times \{y_n\}$. Then the x-axis M is absolutely stable, and isolated from points of generalized recurrence. However, if v is a Liapunov function, $v \mid M = 0$ and $v(x\pi t) \to 0$ as $t \to \infty$ for x in a positively invariant neighborhood of M, then $v(z_n) = 0$ for large n, so that $M \neq v^{-1}(0)$.

6. Absolute Stability: Class Theory

Proposition 20. *In a phase space X, let $\mathcal{M}$ be any collection of subsets of X with the following two properties:*

20.1. *Each $M \in \mathcal{M}$ is closed and positively invariant.*

20.2. *Every $M \in \mathcal{M}$ is the intersection of neighborhoods of M which belong to $\mathcal{M}$.*

Then every member of $\mathcal{M}$ is absolutely stable.

Proof. We will show, by induction on ordinals $\alpha \geqslant 1$, that $D_\alpha^+(M) \subset M$ for all $M \in \mathcal{M}$. The proof for the case $\alpha = 1$ (i.e., $D^+(M) \subset M$ for $M \in \mathcal{M}$) is practically the same as the inductive step; so we will present only the latter.

Assume that $D_\lambda^+(M) \subset M$ for all ordinals $\lambda < \alpha$ and all $M \in \mathcal{M}$; and take some M in $\mathcal{M}$. By assumption, $M = \cap M_j$ where $M_j \in \mathcal{M}$ are neighborhoods of M. To prove $D_\alpha^+(M) \subset \mathcal{M}$, let $x \in D_\alpha^+(y)$ with $y \in M$; recall that D_α^+ is the closure, in $X \times X$, of the union over $\lambda < \alpha$ of $D_\lambda^{+T} = \bigcup_{n=1}^{\infty} D_\lambda^{+n}$. Thus there exist nets $x_i \to x$, $y_i \to y$ with $x_i \in D_\lambda^{+n} y_i$ for suitable $n = n_i$, $\lambda = \lambda_i < \alpha$. Consider any fixed index j; since M_j is a neighborhood of M and $y_i \to y \in M$, ultimately $y_i \in M_j$. By the inductive assumption $D_\lambda^+(M_j) \subset M_j$, we have

$$x_i \in D_\lambda^{+n}(y_i) \subset D_\lambda^{+n}(M_j) \subset M_j ;$$

and since M_j is closed, $x \in \lim x_i \in M_j$. Now, j was arbitrary, so that $x \in \cap M_j = M$, as was to be proved.

THEOREM 21. *Given, a dynamical system in a phase space X which is Hausdorff, paracompact and locally compact. Then the absolutely stable closed subsets of X form the largest class of subsets with properties* 20.1 *and* 20.2.

Proof. From Corollary 16, the class $\mathcal{O}$ of closed absolutely stable subsets of X has properties 20.1 and 20.2; that $\mathcal{O}$ is the largest such class follows from the preceding result.

Remarks. Proposition 20 provides a rather elegant proof for 13.1: the class $\mathcal{M}$ is taken to consist of the sets $\{x \in X : v(x) \leqslant a\}$ for $a \in R^1$ and v a Liapunov function $X \to R^1$.

A similar proof, using Corollary 18 in place of 16, yields the following

THEOREM 22. *In locally compact metrizable phase spaces, the closed absolutely stable subsets form the largest class $\mathcal{M}$, with the following two properties*:

22.1. *Each $M \in \mathcal{M}$ is closed and positively invariant.*

22.2. *Every $M \in \mathcal{M}$ is the intersection of a sequence of neighborhoods of M which belong to $\mathcal{M}$.*

REFERENCES

1. N. P. BHATIA, Stability and Liapunov functions in dynamical systems, *Contr. Diff. Equations* III (1964), 175–188.
2. N. P. BHATIA AND G. P. SZEGÖ, "Dynamical Systems: Stability Theory and Applications," Lecture Notes on Mathematics 35, Springer Verlag, Berlin, 1967.

3. J. Auslander, Generalized recurrence in dynamical systems, *Contr. Diff. Equations* III (1964), 67–74.
4. J. Auslander and P. Seibert, Prolongation and stability in dynamical systems, *Ann. Inst. Fourier (Grenoble)* **14** (1964), 237–268.
5. T. Ura, Sur le courant extérieur à une région invariante, prolongements d'une caractéristique et l'ordre de stabilité, *Funkcial. Ekvac.* **2** (1959), 105–143.
6. L. Nachbin, "Topology and Order," Van Nostrand, Princeton, N. J., 1965.
7. J. Dugundji, "Topology," Allyn and Bacon, Boston, 1966.
8. J. L. Kelley, "General Topology," Van Nostrand, Princeton, N. J., 1955.